ON THE PROPERTIES OF GENERALIZED HARMONIC AND OSCILLATORY NUMBERS. SIMPLE PROOF OF THE PRIME NUMBER THEOREM

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Abstract. We derived the sum identities for generalized harmonic and corresponding oscillatory numbers for which a sieve procedure can be applied. The obtained results enable us to understand better the properties of these numbers and their asymptotic behavior. On the basis of these identities a simple proof of the Prime Number Theorem is represented.

Keywords: generalized harmonic number, oscillatory number, sieve procedure, Möbius inversion, distribution of primes, the Prime Number Theorem

1. Generalized harmonic numbers

In our earlier report we have discussed the regular parts for basic functions of prime numbers [1]. Before considering their oscillatory parts, we would like to discuss some important properties of the generalized harmonic and corresponding oscillatory numbers.

The generalized harmonic number in power s is given by

$$H_x(s) = \sum_{k=1}^{x} \frac{1}{k^s}, \tag{1}$$

where s is any complex number. The basic properties of these numbers can be found elsewhere [7]. Throughout this paper we use repeatedly Möbius inversion formula [3] and here we give it for references: if for all positive x satisfied

$$\mathcal{G}(x) = \sum_{k=1}^{x} \mathcal{F}\left(\frac{x}{k}\right) \tag{2}$$

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then

$$\mathcal{F}(x)=\sum_{k=1}^{x}\mu(k)\,\mathcal{G}\left(\frac{x}{k}\right) \tag{3}$$

and vice versa, where

$$\mu(n)=\mu_n=\begin{cases}1 & \text{if } n=1,\\ (-1)^m & \text{if } n \text{ is a product of } m \text{ distinct primes,}\\ 0 & \text{if the square of primes divides } n.\end{cases}$$

is Möbius function.

From the definition (1) and Möbius inversion formula (2) and (3) directly follows

$$x^s\cdot H_x(s)=\sum_{k=1}^{x}\left(\frac{x}{k}\right)^s$$

and

$$x^s=\sum_{k=1}^{x}\mu_k\left(\frac{x}{k}\right)^s H_{\frac{x}{k}}(s)\,.$$

Hence we get the important sum identity

$$\sum_{k=1}^{x}\frac{\mu_k}{k^s}H_{\frac{x}{k}}(s)=1. \tag{4}$$

Using Stieltjes integration method, we can rewrite the same equation in integral form

$$\int_{1-}^{x}H_{\frac{x}{y}}(s)\cdot dM_y(s)=1, \tag{5}$$

where $M_y(s)=\sum_{k=1}^{y}\frac{\mu_k}{k^s}$ is corresponding oscillatory number in power s (our notations are similar to those of commonly accepted [2, 4, 5, 6] for the case $s=0$ and $s=1$, see below definitions (35)-(37)).

At s =1 for ordinary harmonic number $H_x(1)\equiv H_x$, we have

$$\sum_{k=1}^{x}\frac{\mu_k}{k}H_{\frac{x}{k}}=1 \tag{6}$$

with corresponding integral form

$$\int_{1-}^{x}H_{\frac{x}{y}}\cdot dM_y=1. \tag{7}$$

Applying in (6) the asymptotic formula for harmonic number

$$H_{\frac{x}{k}}=\sum_{n=1}^{x/k}\frac{1}{n}=\log\left[\frac{x}{k}\right]+\gamma+\frac{1}{2\left[\frac{x}{k}\right]}-\frac{1}{12\left[\frac{x}{k}\right]^2}+\frac{1}{120\left[\frac{x}{k}\right]^4}-...,$$

where $\gamma = 0.5772156\ldots$ is Euler's constant, and substituting $\left[\frac{x}{k}\right]$ approximatly by $\frac{x}{k}$ we obtain

$$\begin{aligned}
&\sum_{k=1}^{x} \tfrac{\mu_k}{k} \log \tfrac{x}{k} + \gamma m_x + \tfrac{1}{2x} M_x - \tfrac{1}{12x^2} M_x(-1) + \ldots \approx 1\,, \\
(8) \quad & m_x \log x - \sum_{k=1}^{x} \mu_k \tfrac{\log k}{k} + \gamma m_x + \tfrac{1}{2x} M_x - \tfrac{1}{12x^2} M_x(-1) + \ldots \approx 1\,, \\
&\sum_{k=1}^{x} \mu_k \tfrac{\log k}{k} \approx -1 + (\log x + \gamma)\, m_x + \tfrac{1}{2x} M_x - \tfrac{1}{12x^2} M_x(-1) + \ldots\,.
\end{aligned}$$

Hence it follows that, if $m_x = o\left(\frac{1}{\log x}\right)$ at $x \to \infty$ (and, as a consequence, all terms on the right from the term with m_x tend to zero), then $\sum_{k=1}^{x} \mu_k \frac{\log k}{k} \to -1$ and vice versa.

For another important case when $s = 0$, $H_x(0) \equiv [x]$, we have well known formula

$$\sum_{k=1}^{x} \mu_k \left[\frac{x}{k}\right] = 1, \tag{9}$$

with corresponding integral form

$$\int_{1-}^{x} \left[\frac{x}{k}\right] \cdot dM_y = 1. \tag{10}$$

Using a sieve procedure, the generalized harmonic number can be expanded onto s-powers of the consecutive prime numbers $2, 3, \ldots, p \le x$ as

$$H_x(s) = 1 + \frac{1}{2^s} H_{\frac{x}{2}}^{②}(s) + \frac{1}{3^s} H_{\frac{x}{3}}^{③}(s) + \ldots + \frac{1}{p^s} H_{\frac{x}{p}}^{ⓟ}(s)\,, \tag{11}$$

where by definition recursively

$$H_x^{ⓟ}(s) \equiv H_x(s) - \frac{1}{2^s} H_{\frac{x}{2}}^{②}(s) - \frac{1}{3^s} H_{\frac{x}{3}}^{③}(s) - \ldots - \frac{1}{p_-^s} H_{\frac{x}{p_-}}^{ⓟ_-}(s)\,, \tag{12}$$

or recurrently

$$
\begin{aligned}
&H_x^{(2)}(s) \equiv H_x(s)\ , \\
&H_x^{(3)}(s) \equiv H_x^{(2)}(s) - \frac{1}{2^s} H_{\frac{x}{2}}^{(2)}(s) = H_x(s) - \frac{1}{2^s} H_{\frac{x}{2}}(s)\ , \\
&H_x^{(5)}(s) \equiv H_x^{(3)}(s) - \frac{1}{3^s} H_{\frac{x}{3}}^{(3)}(s) \\
&\qquad = H_x(s) - \frac{1}{2^s} H_{\frac{x}{2}}(s) - \frac{1}{3^s} H_{\frac{x}{3}}(s) + \frac{1}{6^s} H_{\frac{x}{6}}(s)\ , \\
&\ldots, \\
&H_x^{(p)}(s) \equiv H_x^{(p_-)}(s) - \frac{1}{p_-^s} H_{\frac{x}{p_-}}^{(p_-)}(s)\ ,
\end{aligned} \tag{13}
$$

p_- is the prime preceding the prime p.

Consider asymptotic properties at $x \to \infty$. Let us apply Euler product formula, which is valid for $Re(s) > 1$ [2, 4, 5], to represent the generalized harmonic number limit as

$$
\begin{aligned}
H_\infty(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} &= \left(\sum_{a_2 \geq 0} \frac{1}{2^{a_2 s}}\right) \cdot \left(\sum_{a_3 \geq 0} \frac{1}{3^{a_3 s}}\right) \cdot \left(\sum_{a_5 \geq 0} \frac{1}{5^{a_3 s}}\right) \cdot \ldots \\
&= \prod_{\text{all primes } p} \left(1 - \frac{1}{p^s}\right)^{-1}.
\end{aligned} \tag{14}
$$

Further sieving all even number reciprocals yields

$$
\begin{aligned}
H_\infty^{(3)}(s) = \sum_{(k,2)=1}^{\infty} \frac{1}{k^s} &= \left(\sum_{a_3 \geq 0} \frac{1}{3^{a_3 s}}\right) \cdot \left(\sum_{a_5 \geq 0} \frac{1}{5^{a_3 s}}\right) \cdot \ldots \\
&= \prod_{\text{all primes } p>2} \left(1 - \frac{1}{p^s}\right)^{-1}, \\
\frac{H_\infty^{(3)}(s)}{H_\infty(s)} &= \left(1 - \frac{1}{2^s}\right).
\end{aligned} \tag{15}
$$

Similarly, sieving multiples $p=3$ we have

$$\begin{aligned} H_{\infty}^{⑤}(s) &= \sum_{(k,6)=1}^{\infty} \frac{1}{k^s} = \left(\sum_{a_5 \geq 0} \frac{1}{5^{a_3 s}}\right) \cdot \ldots \cdot \left(\sum_{a_p \geq 0} \frac{1}{p^{a_3 s}}\right) \cdot \ldots \\ &= \prod_{\text{all primes } p>3} \left(1-\frac{1}{p^s}\right)^{-1}, \\ \frac{H_{\infty}^{⑤}(s)}{H_{\infty}(s)} &= \left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right). \end{aligned} \tag{16}$$

Continuing the sieving procedure up to any prime p leads to

$$\begin{aligned} H_{\infty}^{ⓟ}(s) &= \sum_{(k,p_{-}\#)=1}^{\infty} \frac{1}{k^s} = \left(\sum_{a_p \geq 0} \frac{1}{p^{a_3 s}}\right) \cdot \ldots \\ &= \prod_{\text{all primes } p>p_{-}} \left(1-\frac{1}{p^s}\right)^{-1}, \\ \frac{H_{\infty}^{ⓟ}(s)}{H_{\infty}(s)} &= \left(1-\frac{1}{2^s}\right)\cdot\left(1-\frac{1}{3^s}\right)\cdot \ldots \cdot \left(1-\frac{1}{p_{-}^s}\right). \end{aligned} \tag{17}$$

Consider important case $s=1$. The sequence of formula transformations for consecutive prime numbers leads to the following set of identities

$$\begin{aligned} & H_x^{③} = H_x - \tfrac{1}{2} H_{\frac{x}{2}} = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \ldots + \tfrac{1}{q}, \qquad q \leq x \\ & H_x - \tfrac{1}{2} H_{\frac{x}{2}}^{②} = 1 + \tfrac{1}{3} H_{\frac{x}{3}}^{③} + \ldots + \tfrac{1}{p} H_{\frac{x}{p}}^{ⓟ} = H_x^{③}, \\ & \frac{H_x^{③}}{H_x} = \left(1 - \tfrac{1}{2}\frac{H_{\frac{x}{2}}}{H_x}\right), \end{aligned} \tag{18}$$

which limit at $x \to \infty$ is

$$\frac{H_{\infty}^{③}}{H_{\infty}} = \left(1-\frac{1}{2}\right). \tag{19}$$

Similarly for the identities at $p=5$, we write

$$\begin{aligned} & H_x^{⑤} = H_x - \tfrac{1}{2} H_{\frac{x}{2}} - \tfrac{1}{3} H_{\frac{x}{3}} + \tfrac{1}{6} H_{\frac{x}{6}} = 1 + \tfrac{1}{5} + \tfrac{1}{7} + \ldots + \tfrac{1}{q}, \; q \leq x \\ & \frac{H_x^{⑤}}{H_x} = \left(1 - \tfrac{1}{2}\frac{H_{\frac{x}{2}}}{H_x} - \tfrac{1}{3}\frac{H_{\frac{x}{3}}}{H_x} + \tfrac{1}{6}\frac{H_{\frac{x}{6}}}{H_x}\right) \end{aligned} \tag{20}$$

with corresponding limit

$$\frac{H_\infty^{⑤}}{H_\infty} = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right). \tag{21}$$

Continuing the same procedure up to any prime p, we have

$$\frac{H_\infty^{ⓟ}}{H_\infty} = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)...\left(1 - \frac{1}{p_-}\right). \tag{22}$$

Ultimately, sieving all primes p, we obtain

$$\frac{1}{\zeta(1)} = \prod_{\text{all primes p}} \left(1 - \frac{1}{p}\right) = 0, \tag{23}$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{24}$$

is Riemann's zeta function and $\zeta(1) \equiv H_\infty$. Thus we have proved the appropriateness of Euler product formula for $\zeta(s = 1)$.

Consider another important case $s = 0$ and $H_x(0) = [x]$. Using a similar procedure as in (11), $[x]$ can be represented as

$$[x] = 1 + \pi_{\frac{x}{2}}^{②} + \pi_{\frac{x}{3}}^{③} + ... + \pi_{\frac{x}{p}}^{ⓟ}, \tag{25}$$

where by definition recursively

$$\pi_x^{ⓟ} \equiv [x] - \pi_{\frac{x}{2}}^{②} - \pi_{\frac{x}{3}}^{③} - ... - \pi_{\frac{x}{p}}^{p_-}, \tag{26}$$

or recurrently

$$\begin{aligned}
&\pi_x^{②} \equiv [x]\ , \\
&\pi_x^{③} \equiv \pi_x^{②} - \pi_{\frac{x}{2}}^{②} = [x] - \left[\frac{x}{2}\right]\ , \\
&\pi_x^{⑤} \equiv \pi_x^{③} - \pi_{\frac{x}{3}}^{③} = [x] - \left[\frac{x}{2}\right] - \left[\frac{x}{3}\right] + \left[\frac{x}{6}\right]\ , \\
&\cdots \\
&\pi_x^{ⓟ} \equiv \pi_x^{p_-} - \pi_{\frac{x}{p_-}}^{p_-}\ ,
\end{aligned} \tag{27}$$

p_- is the prime preceding p.

Let us consider the case $p = 3$.

$$\pi_x^{③} \equiv [x] - \left[\frac{x}{2}\right] = \sum_{(q,2)=1,\ q\leq x} 1\ ,$$

(28) $$\pi_x^{③} = [x] - \pi_{\frac{x}{2}}^{②} = 1 + \pi_{\frac{x}{3}}^{③} + ... + \pi_{\frac{x}{p}}^{ⓟ},$$

$$\frac{\pi_x^{③}}{x} = \left(\frac{[x]}{x} - \frac{\left[\frac{x}{2}\right]}{x}\right),$$

which limit at $x \to \infty$ is

(29) $$\lim_{x\to\infty} \frac{\pi_x^{③}}{x} = \left(1 - \frac{1}{2}\right).$$

Similarly for the identities at $p = 5$, we write

$$\pi_x^{⑤} \equiv \sum_{(q,6)=1,\ q\leq x} 1\ ,$$

(30) $$\frac{\pi_x^{⑤}}{x} = \left(\frac{[x]}{x} - \frac{\left[\frac{x}{2}\right]}{x} - \frac{\left[\frac{x}{3}\right]}{x} + \frac{\left[\frac{x}{6}\right]}{x}\right)$$

with corresponding limit

(31) $$\lim_{x\to\infty} \frac{\pi_x^{⑤}}{x} = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right).$$

Continuing the same procedure up to any prime p, we have

(32) $$\lim_{x\to\infty} \frac{\pi_x^{ⓟ}}{x} = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) ... \left(1 - \frac{1}{p_-}\right).$$

Thus, from (32) and (18) we can see that

(33) $$\lim_{x\to\infty} \frac{\pi_x^{ⓟ}}{x} = \lim_{x\to\infty} \frac{H_x^{ⓟ}}{H_x},$$

or

(34) $$\lim_{x\to\infty} \pi_x^{ⓟ} \frac{H_x}{xH_x^{ⓟ}} = 1.$$

2. Generalized oscillatory numbers

The same approach as for generalized number $H_x(s)$ can be applied for corresponding oscillatory number in power s

(35) $$M_x(s) = \sum_{k=1}^{x} \frac{\mu_k}{k^s}.$$

Particularly at $s = 1$ and $s = 0$ we have the classic summatory functions [2, 3, 4, 5, 6]

$$M_x(1) \equiv m_x = \sum_{k=1}^{x} \frac{\mu_k}{k}, \tag{36}$$

$$M_x(0) \equiv M_x = \sum_{k=1}^{x} \mu_k \qquad \text{– Mertens' function.} \tag{37}$$

Using Möbius inversion formula (2) and (3) in the same way as for (1), we have

$$x^s \cdot M_x(s) = \sum_{k=1}^{x} \mu_k \left(\frac{x}{k}\right)^s,$$

$$x^s = \sum_{k=1}^{x} \left(\frac{x}{k}\right)^s M_{\frac{x}{k}}(s).$$

From where the analog of identity (4) is obtained in the form

$$\sum_{k=1}^{x} \frac{1}{k^s} \cdot M_{\frac{x}{k}}(s) = 1, \tag{38}$$

also having the integral representation

$$\int_{1-}^{x} M_{\frac{x}{y}}(s) \cdot dH_y(s) = 1. \tag{39}$$

Once again, at $s = 1$ and $s = 0$ we have

$$\sum_{k=1}^{x} \frac{1}{k} \cdot m_{\frac{x}{k}} = 1, \tag{40}$$

$$\int_{1-}^{x} m_{\frac{x}{y}} dH_y = 1 \tag{41}$$

and

$$\sum_{k=1}^{x} M_{\frac{x}{k}} = 1, \tag{42}$$

$$\int_{1-}^{x} M_{\frac{x}{k}} d[y] = 1, \tag{43}$$

respectively.

Obviously, for the equation (38)

$$M_x(s) + \frac{1}{2^s} M_{\frac{x}{2}}(s) + \frac{1}{3^s} M_{\frac{x}{3}}(s) + ... + \frac{1}{[x]^s} M_{\frac{x}{x}}(s) = 1$$

the sieve procedure can be applied. Let us, for example, sieve all s-powers of even numbers (sieving $p = 2$). In this case we rewrite (38) for $x/2$ and multiply both parts of obtained equation by $\frac{1}{2^s}$

$$\frac{1}{2^s} M_{\frac{x}{2}}(s) + \frac{1}{4^s} M_{\frac{x}{4}}(s) + \frac{1}{6^s} M_{\frac{x}{6}}(s) + ... + \frac{1}{2^s} \frac{1}{\left[\frac{x}{2}\right]^s} M_{\frac{(x/2)}{(x/2)}}(s) = \frac{1}{2^s}. \tag{44}$$

Subtracting (44) from (38), we derive

$$M_x(s) + \frac{1}{3^s} M_{\frac{x}{3}}(s) + \frac{1}{5^s} M_{\frac{x}{5}}(s) + ... = 1 - \frac{1}{2^s}. \tag{45}$$

Sieving further $p = 3$, we have

$$M_x(s) + \frac{1}{5^s} M_{\frac{x}{5}}(s) + \frac{1}{7^s} M_{\frac{x}{7}}(s) + ... = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right). \tag{46}$$

Sieve procedure can be continued until any prime p, for which $x \geq p\#$ (primorial) $\equiv 2 \cdot 3 \cdot 5 \cdot ... \cdot p$

$$\begin{aligned} & M_x(s) + \frac{1}{p_{+1}^s} M_{\frac{x}{p_{+1}}}(s) + \frac{1}{p_{+2}^s} M_{\frac{x}{p_{+2}}}(s) + ... \\ & \qquad = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right) ... \left(1 - \frac{1}{p^s}\right), \end{aligned} \tag{47}$$

where p_{+1}, p_{+2}, etc. are all successive primes after prime p up to x.

Ultimately, sieving all primes, we obtain

$$M_\infty(s) = \prod_{\text{all primes p}} \left(1 - \frac{1}{p^s}\right). \tag{48}$$

Hence at $s = 1$ from (48) immediately follows the Prime Number Theorem

$$m_\infty \equiv \sum_{k=1}^{\infty} \frac{\mu_k}{k} = \prod_{\text{all primes p}} \left(1 - \frac{1}{p}\right) = 0. \tag{49}$$

At $s = 0$ the sieve procedure for *M(x)* gives

$$\begin{aligned} & M_x + M_{\frac{x}{2}} + M_{\frac{x}{3}} + ... + M_{\frac{x}{x}} = 1\,, \\ & M_x + M_{\frac{x}{3}} + M_{\frac{x}{5}} + ... = 0\,, \\ & \cdots\,, \\ & M_x + M_{\frac{x}{p_{+1}}} + M_{\frac{x}{p_{+2}}} + ... = 0\,, \end{aligned} \tag{50}$$

for any finite p and $x \geq p\#$. In this case after sieving all primes, we get uncertainty $0 \cdot \infty$ for M_∞.

From Euler product formula, which is valid now at $s = 1$ (14), (23) and (48), follows

$$\varsigma(s) \cdot \vartheta(s) = 1, \tag{51}$$

where we introduced for symmetry

$$\vartheta(s) \equiv M_\infty(s) = \sum_{k=1}^{\infty} \frac{\mu_k}{k^s}. \tag{52}$$

For example at $s = 2$ we have

$$M_\infty(2) = \vartheta(2) = \frac{1}{\varsigma(2)} = \frac{6}{\pi^2}. \tag{53}$$

From Riemann's functional equation

$$\varsigma(s) = 2^s \pi^{s-1} \sin(\pi s/2) \, \Gamma(1-s) \, \varsigma(1-s) \tag{54}$$

and from Euler product follows the functional equations for $\vartheta(s)$-function

$$\vartheta(s) = \left(2^s \pi^{s-1} \sin(\pi s/2) \, \Gamma(1-s)\right)^{-1} \vartheta(1-s). \tag{55}$$

In conclusion, using the approach above, we represent some preliminary results for oscillatory part of Chebyshev's ψ -function

$$\begin{aligned}
&\log[x]! = \sum_{k=1}^{x} \log k = \sum_{k=1}^{x} \psi_{\frac{x}{k}} \ , \\
&\psi(x) = \sum_{k=1}^{x} \mu_k \log\left[\tfrac{x}{k}\right]! \ , \\
&\sum_{k=1}^{x} \psi_{\frac{x}{k}} = \sum_{k=1}^{x} \log\tfrac{k}{x} + [x]\log x \ , \\
&[x]\log x = \sum_{k=1}^{x} \left(\psi_{\frac{x}{k}} + \log\tfrac{x}{k}\right) \ , \\
&\psi_x + \log x = \sum_{k=1}^{x} \mu_k \left[\tfrac{x}{k}\right] \log\tfrac{x}{k} \ .
\end{aligned} \tag{56}$$

Applying the last equation in (56), we can separate the regular and oscillatory parts of the ψ -function

$$\psi_x = x \cdot \sum_{k=1}^{x} \mu_k \frac{1}{k} \log\frac{x}{k} - \left(\sum_{k=1}^{x} \mu_k \left\{\frac{x}{k}\right\} \log\frac{x}{k} + \log x\right), \tag{57}$$

According to (4) and (8) the first term in right hand side tends to x while the second term is oscillatory part, determined by nontrivial zeros of Riemann's zeta function (24) through explicit formula

$$\sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} \approx \sum_{k=1}^{x} \mu_k \left\{ \frac{x}{k} \right\} \log \frac{x}{k} + \log x. \tag{58}$$

More detailed discussions concerning oscillatory parts of the basic functions of prime numbers will be published soon [8].

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